Math of the delta style 3-D robots

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1 Introduction

There are several different types of 3-D printers. The Delta type has a lot of benefits. This article discusses some of the computations that are required for using it.

1.1 The delta system

First it is important to understand how the machine works, and to define the terms that are used throughout this document.

The machine has three vertically moving carriages. These are called u, v and w. The carriages connect with two rods each to the effector: a platform holding the print head, which can be positioned by moving the carriages. Because both rods in each pair are the same length, and they are connected at equal distances on both sides, each pair forms a parallelogram. Because of this, the effector is always horizontal.

For simplification of the computations, the model that is used to describe this system is adjusted in a way that does not change the outcome of the computations:

- The effector is assumed to have zero size. This means that all the rods connect to the same point, which is the effector.
- A pair of rods is assumed to be a single rod. This means that each pair only has one length, which had to be true anyway for the effector to be parallel.

  In practice, if the lengths are measured accurately and they differ slightly, pairs should be chosen to be as close to equal as possible and the length of the pair should be the average of the two.

Positions in space have 3 coordinates and are written as (x, y, z). The origin of the machine is in the center of the build platform. The x axis is parallel to the line through the u and v carriages (when at equal height, and if their radii are equal), where from u to v is the positive x direction. The positive y axis intersects the path of the w carriage. The lines from the origin are orthogonal to the paths of the carriages and make angles of 120° with each other.

The radius of a carriage is the distance from the origin to the path of that carriage.

At the top of the machine are limit switches. They are used to prevent the carriage from moving up past the mechanical limits, and to calibrate its position. The switch position is the distance from the switch to the position of the carriage when the effector is at the origin.

Summarizing: the settings that define a delta machine are three rod lengths, three radii and three switch positions.

2 Moving the effector

When the effector needs to move, the position in space is known, and the carriage positions need to be computed. This is fairly easy:

1. For each carriage, find the horizontal distance from the carriage to the effector target position.
2. Compute the base position of each carriage using $base = \sqrt{rodlength^2 - distance^2}$
3. Add the target z position to all carriage positions.

3 Finding the effector

There are several uses for the reverse: with known carriage positions, the effector position should be computed. For example, this can be used for calibration. The motor positions are measured for a number of positions where the effector is on the build platform, and the machine settings are changed so that at those motor positions the effector is indeed at a z position of 0.

This operation is a lot harder. Figure 1 schematically shows it. The effector is known to be at a fixed distance (the rod length) from each carriage. This means it must be on a sphere of that radius, centered at the carriage. Those are the large transparent spheres in Figure 1. The intersection between two spheres is a circle. The intersection of this circle with the final sphere is two points: one for which the rods are pointing up, and one for which they are pointing down. The down pointing solution is the effector position.

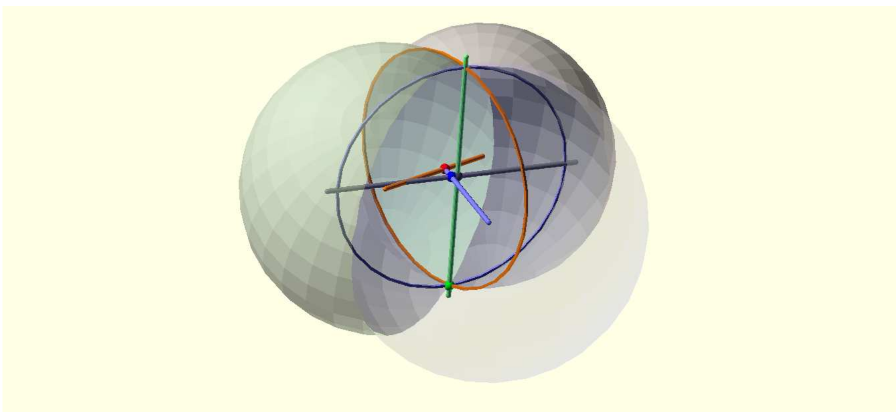

Figure 1: Schematic of the operation of finding the intersection of three spheres.

3.1 Intersection of two spheres

The intersection of two spheres is a circle. This can be defined by its normal and its radius. When viewing a plane through the centers of both spheres, there is a line going straight between the centers, and a line from each center to the intersecting circle. This results in two triangles, as shown in Figure 2.

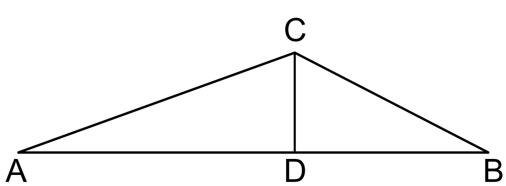

Figure 2: Schematic of the points of interest in the intersections of two spheres. A and B are the sphere centers, C is on the intersecting circle, and D is the center of that circle. The circle itself is perpendicular to the page.

Spheres with centers A and B have radii of AC and BC respectively. D is the center of the intersecting circle. CD is its radius.

For finding the intersecting circle C_{uv}, A and B are known, as are AC and BC. The goal is to find D and CD. Since this is an arbitrary view, C is any point on the circle and cannot be uniquely found. However, given AD as the normal of the circle, D as the center and CD as the radius, the circle is fully defined.

$$
\begin{aligned}
AD^2 &= AC^2 - CD^2 & (1)\\
BD^2 &= BC^2 - CD^2 & (2)\\
AD^2 - BD^2 &= AC^2 - BC^2 & (3)\\
BD &= AB - AD & (4)\\
AD^2 - (AB - AD)^2 &= AC^2 - BC^2 & (5)\\
2AD \cdot AB - AB^2 &= AC^2 - BC^2 & (6)\\
AD &= \frac{AC^2 - BC^2 + AB^2}{2AB} & (7)\\
&= \frac{AC^2 - BC^2}{2AB} + \frac{AB}{2} & (8)\\
D &= A + AD & (9)\\
CD &= \sqrt{AC^2 - AD^2} & (10)
\end{aligned}
$$

This method is first used to find the intersecting circle (orange) of the spheres around u (red) and v (green). The center of this circle (red) is called P_{uv}, the radius is r_{uv} and the normal is n_{uv}.

Then the intersecting circle C_{uvw} (blue) of the sphere around P_{uv} with radius r_{uv} and the sphere around w (blue) is computed. The center of it (blue) is called P_{uvw}, the radius is r_{uvw} and the normal is n_{uvw}.

Because the two intersecting points of circles C_{uv} and C_{uvw} are on both the circles, the line L between them (green) is perpendicular to both their normals. This means that the direction of L can be found using the cross product of n_{uv} and n_{uvw}.

If n_{uv} and n_{uvw} are not orthogonal, L is offset from P_{uvw}. The angle α between n_{uv} and n_{uvw} can be found using the definition of the cosine: $\cos\alpha = \frac{n_{uv} \cdot n_{uvw}}{|n_{uv}| \cdot |n_{uvw}|}$.

The angle between n_{uvw} and the plane of C_{uv} is $90 - \alpha$. The offset must have a length of the tangent of $90 - \alpha$ in the direction orthogonal to both n_{uvw} and L, which can be found using the cross product. In Figure 1 this is shown as the grey line and sphere.

Using the cosine of the inverse sine, the position of the intersection on L can be found. This is the small green sphere in Figure 1.